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CONTROLLABILITY CRITERIA AND SUFFICIENT CONDITIONS FOR DYNAMICAL SYSTEMS TO BE STABILIZABLE†

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Necessary and sufficient conditions for the controllability of non-linear dynamical systems are found. These conditions reduce to the verification of the existence of solutions of partial differential equations of Lyapunov type in stability theory and Levi–Civita type in the theory of invariant manifolds. This governs their utility in stabilization problems and provides a proof of a general theorem relating controllability properties and stabilizability, and extends a previously known theorem for linear systems to the non-linear case. The results are used to investigate the control and stabilizability of the rotational motion of a rigid body by means of a single jet engine.

1. CONTROLLABILITY CRITERIA

We shall study control systems governed by ordinary differential equations

$$\dot{x} = f(x, u) \tag{1.1}$$

where x is the phase vector and u is the control vector. System (1.1) is considered during the time interval $T = [0, \infty)$ on a domain D that is assumed to be a connected n-dimensional C^r-manifold ($r \geqslant 2$). Admissible controls are bounded measurable functions of time $u = u(t)$ taking values in some set $U \subseteq R^m$, and their set is denoted by Ω. We shall also assume that the function $f(x, u)$ is $(r - 1)$ times continuously differentiable with respect to x and u on $D \times \bar{U}$.

We will formulate the controllability criteria in terms of manifolds oriented with respect to the system, and introduced by the following definition [1].

Definition 1. A manifold $K \subset D$ is said to be oriented with respect to system (1.1) in the domain D if it coincides with its positive orbit ($K = \mathrm{Or}^+K$) or negative orbit ($K = \mathrm{Or}^-K$). The positive orbit $K = \mathrm{Or}^+K$ of the set K is the set of points reachable from the set K along trajectories of system (1.1), and the negative orbit Or^-K is the set of those points from which the set K can be reached.

The investigation of oriented manifolds is based on the properties of orbits which in the main are governed by the properties of trajectories. We will formulate some properties of the trajectories of system (1.1) which will be required below. Suppose $u = u(t) \in \Omega$ and that the function $x(t, x_0, u)$ is a solution of the Cauchy problem for system (1.1) with initial condition $x(0) = x_0$ and control $u(t)$. We shall consider the transformation $F_u^t : D \to D$ acting according to the law $F_u^t : x_0 \to x(t, x_0, u)$.

One can show that F_u^t is a diffeomorphism of class C_u^{r-1} (property A).

In many situations in control theory it is sufficient to consider a more restricted set of admissible controls introduced in the following manner. Let V be a denumerable subset of the set U that is dense throughout U. We denote by H the set of piecewise-constant controls taking values in V and switching at rational instants of time. Obviously $H \subset \Omega$. It can be shown that H is a denumerable set.

The following lemma establishes a property of trajectories of system (1.1) which is important when investigating oriented manifolds, which applies when the admissible controls are functions from the set H.

Lemma 1. Suppose $\sup_{u \in U, u \in D} \| f(x, u) \| \leqslant 1$ and $x_1 = f_u^{t_1} x_0$.

Then for any neighbourhood O of the point x_0 one can find $y \in O$, $h \in H$ such that $f_h^{t_1} y = x_1$.

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Theorem 1. System (1.1) is controllable if and only if there are no manifolds N oriented with respect to the system with a smooth boundary such that $N \neq \varnothing, D$.

Necessity is proved by assuming the contrary and using the definition of the controllability property in terms of orbits, which reduces to the fact that $\forall x \in D \ \mathrm{Or}^+x = D$.

Sufficiency is proved in two stages, again assuming the contrary. We shall suppose that unlike the assertion of the theorem, system (1.1) is not controllable, and we shall first show that this reduces to the existence in the domain D of an oriented manifold. We will then show the existence of an oriented manifold with smooth boundary, which completes the proof of the theorem.

By the non-controllability assumption for the system a point x exists in the domain D such that $\mathrm{Or}^+x \neq D$. From the subsets of Or^+x we choose some set γ_0 which is a C^{r-1}-submanifold of D without a boundary of maximal dimension p. Such a set exists, because the set of subsets of Or^+x which are C^{r-1}-submanifolds of D is non-empty and contains, for example, arcs of the trajectories of system (1.1) corresponding to $u = \mathrm{const}$. If $p = 0$ then $\mathrm{Or}^+x = \{x\}$ is a 0-dimensional manifold oriented with respect to system (1.1), and the theorem is proved.

Below we shall assume that $0 < p \leqslant n$. Note that by virtue of property A the sets $F^t_u\gamma_0$ for $t \geqslant 0$, $u \in \Omega$ are p-dimensional C^{r-1}-manifolds without a boundary.

We denote by B_0 the denumerable base of the topology in γ_0 by D, and by Q^+ the set of rational numbers on the semi-axis $[0, \infty)$. We consider the set $B = \{F^t_h\, b_0 \colon b_0 \in B_0, h \in H, t \in Q^+\}$. The set B is denumerable because it is isomorphic to the denumerable set $B_0 \times Q^+ \times H$.

It can be shown that B is a denumerable base for some topology for the set $\mathrm{Or}^+\gamma_0$, and that this topology is Hausdorff.

We shall first establish that $\mathrm{Or}^+\gamma_0 = \cup_{b \in B} b$. Suppose $y_1 \in \mathrm{Or}^+\gamma_0$ and $u_1 \in \Omega$, $t_1 \geqslant 0, y_0 \in \gamma_0$ are such that $y_1 = F^{t_i}_{u_i} y_0$. By Lemma 2 neighbourhoods $0 \subset \gamma_0, z_0 \in 0, h \in H$ exist such that $y_1 = F^{t_i}_{u_i} z_0$. But then, because of the density of Q^+ in $[0, \infty)$ and the absolute continuity of the Lebesgue integral, we find a $b \in B$ such that $y_1 \in b$. Thus $\mathrm{Or}^+\gamma_0 \subset \cup_{b \in B} b$. The opposite inclusion follows from the definition of the set B.

It remains to show that the intersection of any two sets in B is the union of some sets in B, and to do this we show that $\forall y \in \beta = b_1 \cap b_2\, (b_1, b_2 \in B, \beta \neq \varnothing)\, \exists\, b_3 \in B : y \in b_3 \subset \beta$. We represent b_1 and b_2 in the form $b_i = F^{t_i}_{h_i} b^i{}_0$ $(i = 1, 2)$; $t_i \in Q^+, h_i \in H, b_0 \in B_0$. By property A $\beta_0 = (F^{t_i}_{h_1})^{-1}\beta$ is an open subset of γ_0 and $y_0 = (F^{t_i}_{h_1})^{-1} y \in \beta_0 \subset b^1_0$. Because B_0 is a base of the topology of γ_0, we find a $b^3_0 \in B_0$ such that $y_0 \in b_0 \subset \beta_0$, but then $y \in b_3 = F^{t_1}_{h_1} b^3_0 \subset \beta$. The set B is thus a denumerable base for some topology for the set $\mathrm{Or}^+\gamma_0$.

We shall show that this topology is a Hausdorff topology. Indeed, suppose $y_1, y_2 \in \mathrm{Or}^+\gamma_0$ $(y_1 \neq y_2)$ and $y_i \in b_i \in B$. Because D is a Hausdorff space, neighbourhoods O_1 and O_2 of the points y_1 and y_2 exist such that $O_1 \cap O_2 = \varnothing$. We consider the set $O_i \cap b_i$. This is an open set in the topology specified on $\mathrm{Or}^+\gamma_0$ by the base B. This means $\beta_i \in B \colon y_i \in B_i \subset O_i \cap_b$ exist. Then, obviously, $\beta_1 \cap \beta_2 \subset O_1 \cap O_2 = \varnothing$.

We take as the B_0 the coordinate neighbourhoods of some atlas $A_0 = \{(b_0, g_0)\}$ of the C^{r-1}-manifold γ_0. Then the collection $A = \{(F^t_h b_0, g_0(F^t_h)^{-1}) \colon (b_0, q_0) \in A_0, h \in H, t, \in Q^+\}$ is a C^{r-1}-atlas on $\mathrm{Or}^+\gamma_0$. To show this it is sufficient just to verify the consistency of the charts of the atlas A.

Let $b_i = F^{t_i}_{h_i} b^i_0$, $g_i = g^i_0(F^{t_i}_{h_i})^{-1}$, $(b, g_i) \in A$ $(i = 1, 2)$, $b_1 > b_2 \neq \varnothing$. The set $b_1 \cap b_2$ is a p-dimensional submanifold of D. We define the transition map $F \colon R^p \to R^p$ by $\Phi_y = g^1_0(F^{t_1}_{h1})^{-1} F^{t_2}_{h_2}(g^2_0)^{-1}y, y \in R_p$. From the properties of A the map Φ is a C^{r-1}-diffeomorphism. It is obvious that the map id: $\mathrm{Or}^+g_0 \to D$ is an embedding of Or^+g_0 in D, and the set Or^+g_0 is an oriented manifold of system (1.1).

We will use the manifold $\mathrm{Or}^+\gamma_0$ to construct an oriented manifold N with smooth boundary. As a preliminary we note that we can assume the set U, $V_x = \{f(x, u) \colon u \in U\}$ to be compact. If this is not true for the original system (1.1), then we can change to the system $\dot{x} = \varphi(x, u)$, $\varphi(x, u) = f(x, u) \times (1 + f^2(x, u))^{-1/2}$ which is equivalent from the point of view of controllability, and where the set V'_x is bounded with compact closure V_x. We change to a new parameter (to replace u) on the set V_x and obtain a compact bounded set $U(x)$ which will in general depend on the point x.

We shall distinguish between the two cases when $\mathrm{Or}^+\gamma_0$ is manifold without an edge, and when it has an edge. In the first case its dimension is no greater than $n - 1$, and all its points are boundary points, i.e. the boundary is the manifold, which proves the theorem. In the second case a boundary point $\bar{x} \in \mathrm{Or}^+\gamma_0$ exists such that the set $\mathrm{Or}^+\gamma_0$ is situated in some neighbourhood of the point $\bar{x}$ on one side of the plane Π passing through this point. (If there is no such point, then instead of $\mathrm{Or}^+\gamma_0$ we consider $\mathrm{Or}^-\gamma_0$.) We consider the sphere B_s bounded by a sphere $S_0 \subset \mathrm{Or}^+\gamma_0$ of sufficiently small radius, passing through the point $\bar{x}$ and tangential to the plane Π. If the point $\bar{x}$ is a corner point, we consider a point $x_0 \in \mathrm{Or}^+\gamma_0$ that is sufficiently close to it and passing through it a sphere S_0 such that all vectors $f(x_0, u)$ are directed to one side (given by the centre of the sphere) of the tangent plane to the sphere S_0 at the point x_0. We shall show that the boundary $\mathrm{Or}^+ B_s$ ($\mathrm{Or}^- B_s$, if $\mathrm{Or}^+\gamma_0$ is locally concave at the point $\bar{x}$) is a manifold.

For the given choice of sphere S_0 the boundary of $\mathrm{Or}^+ B_s$ is, generally speaking, multiply-connected, and is the image under the transformation F_u^t of several pieces of the sphere S_0. Using the piece δ_0 containing the point $x_0(\bar{x})$ as an example, we shall show that the connected components of the boundary of $\mathrm{Or}^+ B_s$ are manifolds. The set δ_0 is either a manifold, or coincides with the point x_0. In the latter case, instead of the sphere S_0 we take the part of the sphere cut-off by the plane Π, attached at the points of intersection to the plane Π. Then this component of the boundary $\mathrm{Or}^+ B_s$ will also be the image of a manifold, which as before we denote by δ_0.

We note the following property of the boundary points: their pre-images are also boundary points. This follows because of the continuous dependence of the solution on the initial conditions and the image parameter. (Under the transformation F_u^t an internal point becomes an internal point of $\mathrm{Or}^+ B_s$.) Using the compactness property of the sets U and V_x, we leave at the boundary points only those vectors $f(x, u)$, $u \in U'$ which take boundary points into boundary points. Considering the system $\dot{x} = f(x, u)$, $u \in U'$, $x(0) \in \delta_0$ we find that the connected component of the boundary of the set $\mathrm{Or}^+ B_s$ is an orbit of the manifold δ_0 for the given system and, as previously proved, is a manifold, as is $\mathrm{Or}^+\gamma_0$. The theorem is proved.

2. EQUATIONS FOR ORIENTED MANIFOLDS

The orientedness condition means that $\forall u \in U$ the velocity vectors $f(x, u)$ at the boundary points are directed into the exterior of the manifold if $K = \mathrm{Or}^- K$, or into the interior if $K = \mathrm{Or}^+ K$. Suppose the dimension of the manifold is s and its boundary is given locally by the equations $V_i(x) = 0$ ($V_i \in R^1$), with the tangent plane at the point x_0 given by the equations $(x - x_0, \nabla V_i(x_0)) = 0$ ($i = 1, \ldots, n - s$). The interior is given by one of the vectors $\nabla V_i(x_0)$, say $\nabla V_1(x_0)$; here the equations $V_2(x) = 0, \ldots, V_{n-s}(x) = 0$ must be satisfied. It then follows from the orientedness condition that $\forall u \in U$ $(f(x_0, u), \nabla V_i(x_0)) \geqslant 0 (f(x_0, u), \nabla V_i(x_0)) = 0$ ($i = 2, \ldots, n - s$), or $(f(x_0, u), \nabla V_i(x_0)) \leqslant 0$, $(f(x_0, u), \nabla V_i(x_0)) = 0$. These relations can be written in the form of a system of equalities if we introduce the sign-constant function $G(x, u)$ and the continuous functions $\lambda_{ij}(x, u)$ $ij = 1, \ldots, n - s$ in the domain $D \times U$ (dropping the subscript zero from x_0 because of its arbitrariness)

$$
\begin{gathered}
(f(x,u), \nabla V_i(x)) = \sum_{j=1}^{n-s} \lambda_{ij}(x,u) V_j(x) + G_i(x,u) \quad \forall u \in U \\
G_1(x,u) = G(x,u), \quad G_2 = \ldots = G_{n-s} = 0; \quad i = 1, \ldots, n-s.
\end{gathered} \tag{2.1}
$$

These equations were obtained as a consequence of the existence of an oriented manifold for system (1.1). Conversely, if one can find a sign-constant function $G(x, u)$ and continuous functions $\lambda_{ij}(x, u)$ such that the system of equations (2.1) has a solution $V_1(x), \ldots, V_{n-s}(x)$, then system (1.1) has an oriented manifold whose boundary is given by the equations $V_i(x) = 0$ ($i = 1, \ldots, n - s$). Equations (2.1) are obviously satisfied by the function $V_k(x) \equiv 0$ if we put $\lambda_{kj}(x, u) \equiv 0$, $G_k(x, u) \equiv 0$. Hence one can immediately cover all cases if system (2.1) is considered for $s = 1$. An oriented manifold of incomplete dimensionality ($\dim K = s < n$) corresponds to the case when a given number of functions $V_i(x)$ vanish. Using this we obtain another theorem from Theorem 1.

Theorem 2. System (1.1) is controllable if and only if system (2.1) for $s = 1$ has no solutions $V_1(x), \ldots, V_{n-1}(x)$ in D defined by sign-varying functions, for any continuous functions $\lambda_{ij}(x, u)$ and sign-constant functions $G(x, u)$.

Theorem 2 reduces the problem of controlling system (1.1) to that of the existence of a solution of the system of differential equations (2.1). The latter problem is made more complicated by the fact that these equations contain a controlling parameter u which can take any values on the set U. This difficulty can be overcome using a technique similar to the introduction of base systems [2] for constructing invariant manifolds, the essence of which has to do with the fact that for every point $x \in D$ the vector $f(x, u)$ can be represented in the form of a linear combination of vector fields $f_1(x), \ldots, f_k(x)$

$$
\begin{gathered}
f(x,u) = \alpha_1(x,u) f_1(x) + \ldots + \alpha_l(x,u) f_l(x) + \\
+ \alpha_{l+1}(x,u) f_{l+1}(x) + \ldots + \alpha_k(x,u) f_k(x) \quad \forall (x,u) \in D \times U
\end{gathered} \tag{2.2}
$$

where $\alpha_{l+1}(x, u) \geq 0, \ldots, \alpha_k(x, u) \geq 0$, and the coefficients $\alpha_1(x, u), \ldots, \alpha_l(x, u)$ take both positive and negative values. The functions $\alpha_i(x, u), f_i(x)$ $(i = 1, \ldots, k)$ preserve the differential properties of the function $f(x, u)$.

Theorem 3. Suppose system (1.1) is controllable, then the system of equations

$$\left(f_i(x), \nabla V_j(x)\right) = \sum_{\beta=1}^{n-1} \lambda_{ij\beta}(x) V_\beta(x) + G_{ij}(x) \tag{2.3}$$

$$(i = 1, \ldots, k;\ j = 1, \ldots, n-1)$$

where $G_{\beta 1} = G_\beta(x)$ $(\beta = l + 1, \ldots, k)$, and $G_{\alpha\beta} = 0$ for the other index values, has no solutions $V_1(x), \ldots, V_{n-1}(x)$ that are sign-varying functions in the domain D, for any continuous functions $\lambda_{ij\beta}(x)$ and sign-constant functions $G_\beta(x)$.

The proof is performed by assuming the contrary. Suppose a solution $V_1(x), \ldots, V_{n-1}(x)$ of system (2.3) exists for continuous functions $\lambda_{ij\beta}(x)$ and sign-constant functions $G_\beta(x)$. Then this solution will also be a solution of system (2.1) with continuous functions $\lambda_{ij}(x, u) = \Sigma_{\beta=1}^{k} \alpha_\beta(x, u)\lambda_{ij\beta}(x)$ and sign-constant functions $G(x, u) = \Sigma_{i=l+1}^{k} \alpha_i(x, u) G_i(x)$, and by Theorem 2 system (1.1) is not controllable, which contradicts the assertion of the present theorem.

To obtain sufficient conditions it is necessary to investigate the coefficients $\alpha_i(x, u)$ for $(x, u) \in D \times U$ and to study their influence on the behaviour of the trajectories of system (1.1).

3. LOCAL CONTROLLABILITY

In a local formulation the domain D is taken to be some neighbourhood of zero D_0 and it is assumed that the domain U contains the point $u = 0$, and that the function $f(x, u)$ is such that $f(0, 0) = 0$. The property of local controllability is understood in the following way.

Definition 2. System (1.1) is locally controllable (in a neighbourhood of zero) if there exists a neighbourhood of zero $D_{01} \subset D_0$ such that $\forall x_0, x_1 \in D_{01}$ a time $t_1 \in T$ and an admissible control $u(t)$ exist such that the corresponding solution $x(t)$ of system (1.1) satisfies the conditions $x(0) = x_0, x(t_1) = x_1, x(t) \in D_0$ when $0 \leq t \leq t_1$.

Methods developed for stability theory turn out to be useful when investigating local controllability which use Theorem 3. As in stability theory, a function is called sign-constant or sign-varying if a neighbourhood of zero exists in which it preserves its sign or has varying sign, respectively. Using Definition 2 we conclude that non-controllability (in a local sense) can only be the result of the existence of oriented manifolds passing through the origin of coordinates, i.e. the functions $V_1(x), \ldots, V_{n-1}(x)$ defining their boundaries should be sign-varying. Theorem 3 can be reformulated as follows.

Theorem 4. Suppose system (1.1) is locally controllable. Then no sign-varying functions $V_1(x), \ldots, V_{n-1}(x)$ exist that are solutions of system (2.3) for continuous functions $\lambda_{ij\beta}(x)$ and sign-constant functions $G_\beta(x)$.

To investigate the problem of existence for system (2.3) we expand the functions $f_i(x)$ in series in the neighbourhood of zero, and represent the functions $\lambda_{ij\beta}(x)$ and solutions $V_\beta(x)$ in series with undetermined coefficients from those equations in (2.3) where $G_{\alpha\beta} = 0$. The remaining equations in (2.3) determine the sign-constant functions $G_\beta(x)$. These equations must be supplemented with the sign-constancy condition for the functions G_β, and this results in a complete set of relations whose analysis solves the problem of the existence of the functions $V_1(x), \ldots, V_{n-1}(x)$. Additional information about the undetermined coefficients gives a procedure for supplementing the system obtained with Jacobi brackets from Eqs (2.3). As in stability theory, in many cases the local controllability problem can be solved just by looking at the expansion to the second order of smallness.

4. STABILIZABILITY OF NON-LINEAR SYSTEMS

When considering stabilization problems we will take Eqs (1.1) to be the equations of the perturbed motion and we will retain the assumptions made about D $U, f(x, u)$ for the local controllability problem. The stabilization problem is the problem of finding a control $u = u(x(t))$ which ensures the asymptotic stability of the zero solution of the system $\dot{x} = f(x, u(x))$. If such a control exists, then system (1.1) is said to be stabilizable. Note that the control is sought in the form of a function $u(x(t))$, and not in the more general form $u(x(t), t)$, in order that we can use the well-developed techniques of the theory of controlling and stabilizing autonomous systems. For the admissible controls we assume [3] that the functions $u(x)$ are continuously differentiable and that $u(0) = 0$.

To prove a theorem on stabilizability the following auxiliary result is required.

Lemma 2. System (2.1) does not have a solution $V_i(x)$ $(i = 1, \ldots, n-s)$ expressed in terms of sign-varying functions, $\forall u \in U$ if and only if this solution does not exist for some admissible control $u(x)$.

The sufficiency is obvious. We will prove necessity by assuming the opposite. Suppose that for some admissible control $u(x)$ one can find continuous functions $\lambda_{ij}^u(x)$ and a positively-constant function $G^u(x)$ such that the sign-varying functions $V_i^u(x)$ $(i = 1, \ldots, n-s)$ are a solution of the system

$$(f(x,u(x)), \nabla V_i^u(x)) = \sum_{j=1}^{n-s} \lambda_{ij}^u(x) V_j^u(x) + G_i^u(x)$$
$$G_1^u(x) = G^u(x), \; G_2 = \ldots = G_{n-s} = 0, \; i = 1, \ldots, n-s \tag{4.1}$$

Then for any arbitrarily chosen admissible control $u(x)$ a function $V(x)$ exists which gives a solution $V_1 = V(x)$, $V_2 = \ldots = V_{n-1} = 0$ of system (4.1) such that for all x and u in a sufficiently small neighbourhood $D_0 \times U$ of the point (0, 0) the vectors $f(x, u)$ for $u \in U$ are situated on one side of the tangent plane to the surface $V(x) = 0$ at the point $x \in D_0$. If such a surface does not exist, then by the smoothness of the vector field $f(x, u)$ one could choose a continuously-differentiable function $u_n(x)$ in such a way that for $u_n(x)$ no surface exists with respect to which the vectors $f(x, u)$ lie only on one side, i.e. Eqs (4.1) would not have a solution for the given $u_n(x)$. In view of this the function $V(x)$ determines a solution of system (2.1) defined in some neighbourhood D_0 for all $u \in U$, which contradicts the condition and proves the lemma.

Theorem 4, with the Krasovskii instability theorem [4] and the Barbashin–Krasovskii theorem on asymptotic stability [5], enable us to prove the following result for linear systems.

Theorem 5. If system (1.1) is locally controllable, it is stabilizable.

Proof. According to Theorem 4, the local controllability means that for any continuous functions $\lambda_{ij}(x, u)$ and sign-constant functions $G_i(x, u)$ there is no solution of system of equations (2.1) expressible in terms of sign-varying functions $V_1(x), \ldots, V_{n-s}(x)$ $\forall u(x) \in U$. Then by Lemma 2 a control $u = u(x)$ exists such that the system of equations for $u = u(x)$ has no solution. In particular, for $i = 1$ the equation

$$(f(x, u(x)), \; \nabla V_1(x)) = \lambda(x, u(x)) V_1(x) + G(x, u(x))$$

has no solution expressible in terms of a sign-varying function, for any continuous function $\lambda(x, u(x))$ and sign-constant function $G(x, u(x))$, which includes functions $\lambda(x, u(x))$ such that $\lambda > 0$. From this we conclude from the Krasovskii instability theorem [4] that the zero solution is not unstable, i.e. it is stable. This means a positive-definite Lyapunov function exists with a negatively constant derivative.

By the Barbashin–Krasovskii theorem on asymptotic stability [5] the zero solution is not just stable, but also asymptotically stable, because for the chosen control no whole semi-trajectories exist passing through zero (including the set of trajectories on which the derivative of the Lyapunov function vanishes). This follows from the fact that the trajectories passing through zero are given by the vanishing of the sign-varying functions $V_1(x), \ldots, V_{n-1}(x)$ which are the solution of system (2.1) when $u = u(x)$, $G(x, u(x)) = 0$ and some continuous functions $\lambda_{ij}(x, u(x))$, which cannot be true by the condition of the theorem. The theorem is proved.

5. CONTROL OF A ROTATING RIGID BODY

Many problems on the motion of a rigid body about its centre of mass under the influence of a thrust are studied by using as a model an absolutely rigid body with no mass variation. The equations of motion have the form

$$\dot{\omega}_1 = a_1 \omega_2 \omega_3 + \alpha_1 u \;\; (1\;2\;3) \tag{5.1}$$

where $a_1 = (A_2 - A_3)/A_1$, $\alpha_1 = e_1/A_1$ (1 2 3), A_1, A_2, A_3 are the principal central moments of inertia of the body, ω_1, ω_2, ω_3 are projections of the angular velocity vector ω onto the principal central axes, $e = (e_1, e_2, e_3)$ is the unit vector directed along the thrust, and u is the control which describes the magnitude of the thrust.

We will investigate, using Theorem 3.5, the controllability and stabilizability of system (5.1). System (5.1) has the obvious representation $\dot{\omega} = uf_1(\omega) + f_2(\omega)$ which we shall use to obtain Eqs (2.3).

We begin the investigation of system (2.3) with the case $V_1 = V$, $V_2 = 0$. We have

$$\begin{aligned} &L_1 = \alpha_1 p_1 + \alpha_2 p_2 + \alpha_3 p_3 - \lambda_1 V = 0, \quad p_1 = \partial V / \partial \omega_1 \quad (1\ 2\ 3) \\ &L_2 = a_1\omega_2\omega_3 p_1 + a_2\omega_3\omega_1 p_2 + a_3\omega_1\omega_2 p_3 - \lambda_2 V - G = 0 \end{aligned} \tag{5.2}$$

We complete system (5.2) with an equation obtained from the vanishing of the Jacobi bracket of the operators L_1 and L_2 which is calculated using the equalities $L_1 = 0$. $L_2 = 0$

$$\begin{aligned} &L_3 = [L_1, L_2] = a_1(\alpha_2\omega_3 + \alpha_3\omega_2)p_1 + a_2(\alpha_3\omega_1 + \\ &+\alpha_1\omega_3)p_2 + a_3(\alpha_1\omega_2 + \alpha_2\omega_1)p_3 - \lambda_3 V - G_1 = 0 \end{aligned} \tag{5.3}$$

In Eqs (5.2) and (5.3) λ_1, λ_2, λ_3, G, G_1 are functions of the variables ω_1, ω_2, ω_3. The determinant of system (5.2), (5.3), considered as a system of linear algebraic equations for p_1, p_2, p_3 is equal to

$$\begin{aligned} &\Delta = \alpha_1 a_2 a_3(\alpha_2\omega_3 - \alpha_3\omega_2)\omega_1^2 + \alpha_2 a_3 a_1(\alpha_3\omega_1 - \\ &-\alpha_1\omega_3)\omega_2^2 + \alpha_3 a_1 a_2(\alpha_1\omega_2 - \alpha_2\omega_1)\omega_3^2 \end{aligned} \tag{5.4}$$

If $\Delta \neq 0$, the solution of system (5.2), (5.3) has the form [2] $V = c \exp \psi(\omega_1, \omega_2. \omega_3)$ and is not a sign-varying function. If $\Delta = 0$, the manifold defined by this equation is two-dimensional, and an oriented manifold of full dimension with boundary $V = 0$ does not exist. Thus the conditions of Theorem 3 are satisfied in this case.

We now consider the case $V_1 = V$, $V_2 = W$. System (2.3) takes the form

$$\begin{aligned} &L_1 = \alpha_1 p_1 + \alpha_2 p_2 + \alpha_3 p_3 - \lambda_1 V - \lambda_{12} W = 0, \quad p_1 = \partial V / \partial \omega_1 \quad (1\ 2\ 3) \\ &L_2 = a_1\omega_2\omega_3 p_1 + a_2\omega_3\omega_1 p_2 + a_3\omega_1\omega_2 p_3 - \lambda_{21} V_1 - \lambda_{22} W - G = 0 \\ &L_4 = \alpha_1 q_1 + \alpha_2 q_2 + \alpha_3 q_3 - \lambda_{41} V - \lambda_{42} W = 0, \quad q_1 = \partial W / \partial \omega_1 \quad (1\ 2\ 3) \\ &L_5 = a_1\omega_2\omega_3 q_1 + a_2\omega_3\omega_1 q_2 + a_3\omega_1\omega_2 q_3 - \lambda_{51} V - \lambda_{52} W = 0 \end{aligned} \tag{5.5}$$

Completing this system with the equations $L_3 = [L_1, L_2] = 0$, $L_6 = [L_4, L_5] = 0$ and considering the resulting system $L_i = 0$ $(i = 1, \ldots, 6)$ as a system of linear algebraic equations in p_i, q_i $(i = 1, 2, 3)$, we find that its determinant is equal to Δ^2. As before, we conclude that the conditions of Theorem 3 are satisfied if $\Delta \neq 0$. To complete the analysis we need to check whether the set given by condition $\Delta = 0$ contains an invariant manifold common to the base systems

$$\dot{\omega}_1 = \alpha_1, \quad \dot{\omega}_2 = \alpha_2, \quad \dot{\omega}_3 = \alpha_3 \tag{5.6}$$

$$\dot{\omega}_1 = a_1\omega_2\omega_3, \quad \dot{\omega}_2 = a_2\omega_3\omega_1, \quad \dot{\omega}_3 = a_3\omega_1\omega_2 \tag{5.7}$$

We begin the investigation with system (5.6). We compute the derivatives of Δ using system (5.6), having first transformed the determinant (5.4). We find that

$$\begin{aligned} &\Delta = \alpha_2\alpha_3\omega_1 n_1 + \alpha_3\alpha_1\omega_2 n_2 + \alpha_1\alpha_2\omega_3 n_3 \\ &\dot{\Delta}_{(5.6)} = -2(\alpha_1 s_1\omega_2\omega_3 + \alpha_2 s_2\omega_3\omega_1 + \alpha_3 s_3\omega_1\omega_2) \\ &\ddot{\Delta}_{(5.6)} = 2(\alpha_2\alpha_3 s_1\omega_1 + \alpha_3\alpha_1 s_2\omega_2 + \alpha_1\alpha_2 s_3\omega_3) \end{aligned} \tag{5.8}$$

where $n_1 = a_1(a_3\omega_2^2 - a_2\omega_3^2)$, $s_1 = a_1(a_3\omega_2^2 - a_2\omega_3^2)$ (1 2 3).

By the method of invariant relations [2] an invariant manifold of system (5.6) contained in the set specified by condition $\Delta = 0$ is found by solving the system of equations obtained by the vanishing of the derivatives $\dot{\Delta}_{(5,6)}$ and $\ddot{\Delta}_{(5,6)}$. This system admits of two classes of solution

$$\omega_1 = 0 \text{ when } \alpha_1 = 0\ (1\ 2\ 3) \tag{5.9}$$

$$\alpha_1\omega_3 - \alpha_3\omega_1 = 0 \text{ when } s_2 = 0\ \ (1\ 2\ 3) \tag{5.10}$$

To establish this fact it is convenient to change to the variables $x = \omega_2/\omega_1, y = \omega_3/\omega_1$. Then the system takes the form

$$\begin{aligned} &\alpha_3\alpha_1 s_2 x + \alpha_1\alpha_2 s_3 y = -\alpha_2\alpha_3 s_1 \\ &\alpha_1 s_1 xy + \alpha_2 s_2 y + \alpha_3 s_3 x = 0 \\ &\alpha_2\alpha_3\tilde{n}_1 + \alpha_3\alpha_1 x\tilde{n}_2 + \alpha_1\alpha_2 y\ \ \tilde{n}_3 = 0 \end{aligned} \tag{5.11}$$

where $\tilde{n}_1 = a_1(a_3x^2 - a_2y)$, $\tilde{n}_2 = a_2(a_1y^2 - a_3)$, $\tilde{n}_3 = a_3(a_2 - a_1x^2)$.

From the first two equations in (5.11) we find the values of x and y and substitute them into the last equation, using the equalities $s_1 + s_2 + s_3 = 0, \tilde{n}_1 + \tilde{n}_2 + \tilde{n}_3 = 0$.

We now consider whether the base system (5.7) has solutions (5.9) and (5.10). Substituting solution (5.9) into Eqs (5.7) we find that $\omega_1 = 0$ is an invariant relation of system (5.7), given the condition $a_1 = 0$ or when we have the additional relation $\omega_2 = 0$. In order for $\omega_2 = 0$ to be an invariant relation of system (5.6) it is necessary for the additional condition $\alpha_2 = 0$ to be satisfied, which is verified by direct substitution into (5.6). To verify that $\varphi = \alpha_1\omega_3 - \alpha_3\omega_1 = 0$ is an invariant relation of system (5.7), using $s_2 = 0$ we find $\dot{\varphi}_{(5.7)} = -\alpha_3 a_1\omega_2\varphi/\alpha_1 = 0$, i.e. $\varphi = 0$ defines an invariant manifold of system (5.7) without additional restrictions on the parameters.

We finally conclude that the set specified by the condition $\Delta = 0$ contains an invariant manifold common to the base systems (5.6) and (5.7), under the following conditions

$$\begin{aligned} &(1)\ \alpha_1 = \alpha_2 = 0\ (1\ 2\ 3) \\ &(2)\ \alpha_1 = 0,\ \ a_1 = 0\ (1\ 2\ 3) \\ &(3)\ a_1\alpha_3^2 - a_3\alpha_1^2 = 0\ (1\ 2\ 3) \end{aligned} \tag{5.12}$$

We indicate the solution of system (5.5) for each of the cases (5.12)

$$\begin{aligned} &(1)\ V = \omega_1,\ \ W = \omega_2\ \ (G = 0,\ \lambda_{ij} = 0) \\ &(2)\ V = \omega_1,\ \ W = 0\ \ (G = 0,\ \lambda_{ij} = 0) \\ &(3)\ V = \alpha_1\omega_3 - \alpha_3\omega_1,\ \ W = 0 \\ &(G = 0,\ \lambda_{ij} = 0, \text{ except for } \lambda_{21} = -\ \alpha_3 a_1\omega_2/\alpha_1) \end{aligned}$$

Thus, if the parameters of system (5.1) do not satisfy conditions (5.12), then the conditions of Theorem 3 are satisfied by system (5.1), and so the latter is controllable. Moreover, by Theorem 5 it is also stabilizable. When conditions (5.12) are satisfied system (5.1) is non-controllable.

Note that the controllability of system (5.1) has been previously considered [2, 6, 7] in a similar formulation, and that the case of dynamical symmetry of the rigid body was excluded in [6] because it required special considerations. In this paper that case has not been excluded and has been analysed in the same way as for an asymmetric rigid body.

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